Black-Box Anisotropic Mesh Generators for Engineering Applications

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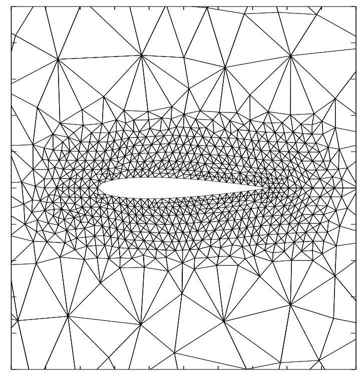
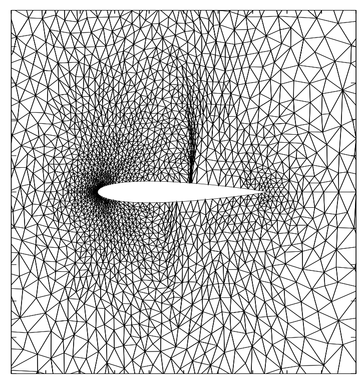
The adaptive mesh methods significantly improve accuracy of simulations and allow to solve large problems appearing in engineering applications. The majority of these methods use meshes with regular shaped elements. The wide usage of anisotropic meshes was retarded by two barriers. First, it was proved in [1] that the presence of acute angles causes deterioration of finite element discretizations of piece-wise polynomial functions with isotropic second derivatives. Second, there were not enough robust methods for generation unstructured anisotropic meshes. The methodology developed in [2-10] gives different perspective on anisotropic mesh generation methods.

It was shown in [2, 9] that simplexes with acute angles stretched along the direction of minimal second derivative of a solution are the best elements for minimizing the interpolation error. Thus, such simplexes might be and should be used in the regions where solution Hessian (matrix of second derivatives) is anisotropic. Recently, a few robust methods using the solution Hessian for generating adaptive anisotropic meshes were proposed (see [4, 3] and references therein). The theoretical analysis of these methods based was done in [3,5-8,10] for piecewise linear discretizations. The optimal error estimates in L_∞ norm have been proved there for the interpolation error.

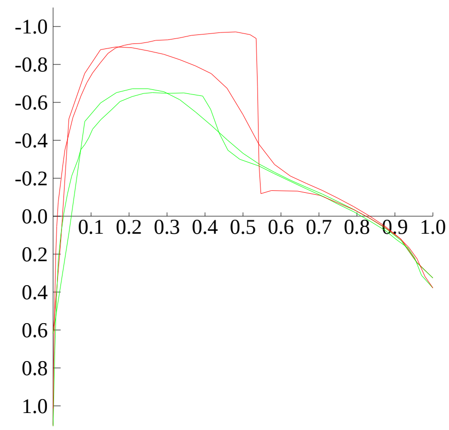
The idea of the new methodology is to generate a mesh which is quasi-uniform in a metric $|H|$ generated by the solution Hessian H. The measure of the quasi-uniformity, $0 \le Q(|H|, \Omega_h) \le 1$, is a function of the metric and the mesh and is called the *mesh quality*. The anisotropic mesh is generated by a sequence of local mesh modifications of a simple initial mesh. This increases *robustness* of the methodology, allows its *black box* implementation [6] and simplifies *control* of the adaptation process [7, 8]. The appeal of such a methodology for solving engineering problems is difficult to overestimate.



A zoom of the initial isotropic mesh $\Omega_h^{(1)}$.



A zoom of the final anisotropic mesh $\Omega_h^{(10)}$.



Pressure profiles along the top (red) and bottom (green) sides of the wing. The sharper profiles correspond to the final mesh.

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Let us consider a model 2D problem: compressible irrotational isentropic adiabatic flow of the ideal gas around a wing. The unknown u is the potential of the gas velocity. This problem is strongly non-linear and the existence of a solution is not guaranteed mathematically. The solution algorithm is presented below.

1. Generate an initial triangulation $\Omega_h^{(1)}$.
2. For $k = 1, 2, \ldots$ repeat:
   (a) Find a discrete u_h on mesh $\Omega_h^{(k)}$.
   (b) Compute the discrete Hessian H of u_h.
   (c) Terminate adaptive loop if the mesh quality satisfies
   $$Q(|H|, \Omega_h^{(k)}) > Q_0$$
   where Q_0 is a constant ($Q_0 \sim 0.2 - 0.7$).
   (d) Generate the next mesh $\Omega_h^{(k+1)}$ such that
   $$Q(|H|, \Omega_h^{(k+1)}) \geq Q_0.$$

The initial and the final meshes are shown on the previous page. The final mesh has approximately the same number of triangles as the initial mesh but results in more accurate pressure profiles along both sides of the wing.

It is pertinent to note that the initial mesh can be very coarse. Its further refinement and adaptation does not require user's intervention and is performed in a very robust manner using local mesh modifications. Only the modifications which are topologically acceptable and increase the mesh quality are performed. The list of modifications includes the following operations (see [10] for more detail).

- Put a new point in the middle of a mesh edge and split each element sharing the edge into two elements.
- Remove the common edge (face in 3D) of two elements and connect opposite vertices by a new edge.
- Remove a mesh edge and split thus appearing quadrilateral (or polyhedron in 3D) into a new set of simplices.
- Delete a mesh node together with all mesh edges ending in it and split thus appearing quadrilateral (or polyhedron in 3D) into a new set of simpleces.
- Move a mesh node inside a super-element consisting of elements sharing the node.

The methodology can be generalized to problems with many primary variables.

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